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LETTER TO THE EDITOR

Some multidimensional integrals related to many-body systems with the $1/r^2$ potential

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Abstract. An N-dimensional integral evaluated by K Aomoto is shown to represent the density matrix for an impurity particle in the $1/r^2$ quantum many-body problem on a line. The value of the N-dimensional integral representing the same density matrix in periodic boundary conditions is conjectured, as is the value of an N-dimensional integral which represents a two-point correlation function in the system. Also, the partition function of a related classical Hamiltonian is evaluated by formulating a conjecture which asserts that the sum of Jacobians of a certain change of variables in N-dimensions is a constant.

Aomoto [1] has recently obtained a closed form evaluation of some two-point correlations with respect to the measure

$$(p_{1,\lambda}(x_1, x_2, \ldots, x_N))^2 \, dx_1 \ldots dx_N \tag{1a}$$

where

$$p_{a,\lambda} = \exp\left(-\frac{a}{4}(x_1^2 + \ldots + x_N^2)\right) \prod_{1 \leqslant j < k \leqslant N} |x_k - x_j|^{\lambda/2}. \tag{1b}$$

In particular, the correlation

$$f(y_1, y_2) = \left(\prod_{\ell=1}^{N} \int_{-\infty}^{\infty} dx_\ell (y_1 - x_\ell)(y_2 - x_\ell)\right)\left(p_{1,\lambda}(x_1, \ldots, x_N)\right)^2 \tag{2}$$

was evaluated, and the corresponding asymptotic formula for $f(\xi_1/\sqrt{2N}, \xi_2/\sqrt{2N})$ given in the large-N limit with ξ_1 and ξ_2 fixed. In this letter we will provide a physical interpretation of (2) in terms of the one-point density matrix of a certain many-body quantum system, and use the asymptotic formula to obtain the corresponding momentum distribution. The value of the N-dimensional integral representing the same density matrix in periodic boundary conditions is conjectured, as is the value of another N-dimensional integral which represents a two-point correlation function in the system. In addition, we consider the classical partition function of a system related to the quantum Hamiltonian and obtain its evaluation by using a conjectured identity regarding a change of variables in N-dimensions.

We begin with the following observation:

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Proposition 1. The function

$$\psi_{a,\lambda}(y;x_1,\ldots,x_N)=\exp(-ay^2/2\lambda)\prod_{\ell=1}^{N}(y-x_\ell)p_{a,\lambda}(x_1,\ldots,x_N) \tag{3}$$

is an exact wavefunction for the Hamiltonian

$$H=-\sum_{j=1}^{N}\frac{\partial^2}{\partial(x_j)^2}-\frac{\lambda}{2}\frac{\partial^2}{\partial y^2}+a^2/4\sum_{j=1}^{N}(x_j)^2+(a^2/2\lambda)y^2 \\ +g\sum_{1\leqslant j<k\leqslant N}\frac{1}{(x_k-x_j)^2} \tag{4}$$

where λ and g are related by

$$\lambda=1+(1+2g)^{\frac{1}{2}}. \tag{5}$$

This system corresponds to identical quantum particles with coordinates $x_1,\ldots,x_N$ in a harmonic well interacting via a $1/r^2$ pair potential, and a single impurity particle with coordinate y also in a harmonic well (of different strength) which does not interact with the other particles. The mass of the impurity particle to the other particles is in the ratio $(\lambda/2):1$.

Proposition 1 is a special case of a more general result for an exact wavefunction of the t-species Hamiltonian

$$H_t=-\sum_{\alpha=1}^{t}\frac{1}{m_\alpha}\sum_{j=1}^{N_\alpha}\frac{\partial^2}{\partial(x_j^{(\alpha)})^2}+\sum_{\alpha=1}^{t}g_\alpha\sum_{1\leqslant j<k\leqslant N_\alpha}\frac{1}{|x_k^{(\alpha)}-x_j^{(\alpha)}|^2} \\ +\sum_{1\leqslant\alpha<\beta\leqslant t}g_{\alpha\beta}\sum_{j=1}^{N_\alpha}\sum_{k=1}^{N_\beta}\frac{1}{|x_j^{(\alpha)}-x_k^{(\beta)}|^2}+a^2\sum_{\alpha=1}^{t}m_\alpha\sum_{j=1}^{N_\alpha}(x_j^{(\alpha)})^2. \tag{6}$$

Theorem 1. The function

$$\exp\left[-\frac{a}{2}\sum_{\alpha=1}^{t}m_\alpha\sum_{j=1}^{N_\alpha}(x_j^{(\alpha)})^2\right]\prod_{\alpha=1}^{t}D(x^{(\alpha)})\prod_{1\leqslant\alpha<\beta\leqslant t}D(x^{(\beta)},x^{(\alpha)}) \tag{7a}$$

where

$$D(x^{(\alpha)})=\prod_{1\leqslant j<k\leqslant N_\alpha}|x_k^{(\alpha)}-x_j^{(\alpha)}|^{(m_\alpha)^2} \tag{7b}$$

and

$$D(x^{(\beta)},x^{(\alpha)})=\prod_{j=1}^{N_\beta}\prod_{k=1}^{N_\alpha}|x_k^{(\alpha)}-x_j^{(\beta)}|^{m_\alpha m_\beta} \tag{7c}$$

is an exact wavefunction of the Hamiltonian (6) provided

$$g_\alpha=2m_\alpha(m_\alpha^2-1)\qquad\text{and}\qquad g_{\alpha\beta}=(m_\alpha+m_\beta)(m_\alpha m_\beta-1). \tag{8}$$

Remarks. (1) As presented (7) corresponds to the ground state wavefunction since it is nodeless for finite values of the potential. However, for $m_\alpha m_\beta$ odd the absolute value signs can be removed and (7) still satisfies the Schrödinger equation with Hamiltonian (6) (proposition 1 is of this type). The wavefunction no longer corresponds to the ground state.

(2) An exact wavefunction corresponding to the periodic version of (6) has been given by Krivnov and Ovchinnikov [2] (see also [3]).

The key to theorem 1 is the identity [4]

$$\frac{1}{(x-y)(x-z)}+\frac{1}{(y-x)(y-z)}+\frac{1}{(z-x)(z-y)}=0 \tag{9}$$

which shows that the apparent three-body terms which result from applying the kinetic energy operator of (6) to (3) cancel. We find that the energy eigenvalue corresponding to the wavefunction (7) is

$$a\left[\sum_{\alpha=1}^{t}N_\alpha+\sum_{\alpha=1}^{t}(q_\alpha)^2N_\alpha(N_\alpha-1)+\sum_{\substack{\alpha=1,\gamma=1\\ \alpha\neq\gamma}}^{t}q_\alpha q_\gamma N_\alpha N_\gamma\right]. \tag{10}$$

In the thermodynamic limit the one-particle density matrix $\rho(y_1,y_2)$ for the impurity particle of the Hamiltonian (4) in the state (3) is defined as

$$\rho(y_1,y_2)=\lim_{N\to\infty}\frac{\prod_{\ell=1}^{N}\int_{-\infty}^{\infty}\mathrm{d}x_\ell\,\psi_{a,\lambda}(y_1;x_1,\ldots,x_N)\psi_{a,\lambda}(y_2;x_1,\ldots,x_N)}{\prod_{\ell=1}^{N}\int_{-\infty}^{\infty}\mathrm{d}x_\ell(\psi_{a,\lambda}(y_1;x_1,\ldots,x_N))^2} \tag{11}$$

where the normalization has been chosen so that $\rho(y,y)=1$. In the thermodynamic limit we require the identical particles to tend to a finite density, η say. This can be achieved [5, equation (21)] by choosing

$$a=\lambda(\pi\eta)^2/2N. \tag{12}$$

Changing variables $\sqrt{a}x_\ell=X_\ell$ in (11) and use of (12) and the definitions (3) and (2) gives

$$\rho(y_1,y_2)=\lim_{N\to\infty}f(\sqrt{\lambda/2N}\pi\eta y_1,\ \sqrt{\lambda/2N}\pi\eta y_2)/f(\sqrt{\lambda/2N}\pi\eta y_1,\sqrt{\lambda/2N}\pi\eta y_1). \tag{13}$$

The value of (13) can be read off from Aomoto's result [1, theorem of section 4]. We thus obtain

$$\rho(y_1,y_2)=c_\lambda\frac{J_{(2/\lambda)-1/2}(\pi\eta(y_1-y_2))}{[\pi\eta(y_1-y_2)]^{(2/\lambda)-1/2}} \tag{14a}$$

where

$$c_\lambda=2^{(2/\lambda)-1/2}\Gamma((2/\lambda)+1/2) \tag{14b}$$

and $J_p(x)$ denotes the Bessel function of order p and $\Gamma(x)$ denotes the gamma function. The momentum distribution function $n(k)$ is obtained by taking the Fourier cosine transform of $\rho(y_1, y_2) = \rho(y_1 - y_2)$. We find [6]

$$n(k) = \begin{cases} A_\lambda \frac{1}{\pi\eta}(1-(k/\pi\eta)^2)^{2/\lambda-1} & |k| < \pi\eta \\ 0 & |k| > \pi\eta \end{cases} \tag{15a}$$

where

$$A_\lambda = 2\pi^{1/2}\frac{\Gamma(2/\lambda + 1/2)}{\Gamma(2/\lambda)}. \tag{15b}$$

Thus the momentum of the impurity particle lies inside the first Brillouin zone with peaks as $|k| \to \pi\eta$ for $\lambda > 2$ (i.e., from (5), $g > 0$) and at $k = 0$ for $\lambda < 2$ (i.e. $g < 0$). In the special case $\lambda = 2$ the impurity particle becomes indistinguishable from the other particles. The density matrix (14) is then that of non-interacting fermions in one-dimension and (15) gives the correct Fermi surface.

Let us now consider the periodic version of the Hamiltonian (4):

$$H = -\sum_{j=1}^{N}\frac{\partial^2}{\partial(x_j)^2} - \frac{\lambda}{2}\frac{\partial^2}{\partial y^2} + \left(\frac{\pi}{L}\right)^2 g \sum_{1\leqslant j<k\leqslant N}\left(\frac{1}{\sin\pi(x_k - x_j)/L}\right)^2. \tag{16}$$

From [2] we know that

$$\phi_\lambda(y; x_1, \ldots, x_N) = \prod_{j=1}^{N}\sin\pi(y - x_j)/L \prod_{1\leqslant j<k\leqslant N}|\sin\pi(x_k - x_j)/L|^{\lambda/2} \tag{17}$$

is an exact wavefunction of (16) provided λ is given by (5). The density matrix for the impurity particle in the state (17) is proportional to the integral

$$g(y_1, y_2) = \left(\prod_{\ell=1}^{N}\int_0^L \mathrm{d}x_\ell \sin\pi(y_1 - x_\ell)/L \sin\pi(y_2 - x_\ell)/L\right) \times \prod_{1\leqslant j<k\leqslant N}|\sin\pi(x_k - x_j)/L|^\lambda. \tag{18}$$

For $\lambda = 2\gamma$, γ a positive integer, it is straightforward to express $g(y_1, y_2)$ in terms of

$$G_\gamma(Y_1, Y_2) = CT_{\{X_1,\ldots,X_N\}}\prod_{\ell=1}^{N}\left(1 - \frac{Y_1}{X_\ell}\right)\left(1 - \frac{X_\ell}{Y_2}\right)\prod_{j,k=1,j\neq k}^{N}\left(1 - \frac{X_j}{X_k}\right)^\gamma \tag{19}$$

where $CT_{\{X_1,\ldots,X_N\}}$ denotes the 'constant term' with respect to $\{X_1, \ldots, X_N\}$ (i.e. terms independent of $\{X_1, \ldots, X_N\}$) in the multivariable Laurent expansion of the products. Using the van der Monde determinant expansion it is easy to show that

$$G_1(Y_1, Y_2) = N!\sum_{k=0}^{N}\left(\frac{Y_1}{Y_2}\right)^k. \tag{20}$$

On the basis of (20) and exact numerical data for small N we propose the following exact evaluation of (19) for general $\gamma \in \mathbb{Z}^+$.

Conjecture 1.

$$G_\gamma(Y_1, Y_2) = c_\gamma \sum_{k=0}^{N} \frac{(-N)_k(1/\gamma)_k}{k!(-N-1/\gamma+1)_k} \left(\frac{Y_1}{Y_2}\right)^k$$
$$= c_\gamma \ {}_2F_1(-N, 1/\gamma; -N-1/\gamma+1; Y_1/Y_2) \tag{21}$$

where

$$c_\gamma = (\gamma N)!/(\gamma!)^N \tag{22}$$

and ${}_2F_1(a, b; c; x)$ denotes the hypergeometric function of Gauss.

Remarks. (1) The value of c_γ is given by the so-called Dyson conjecture [7].
(2) The conjecture (21) is a reciprocal polynomial in Y_1/Y_2, which is required from the definition (19).
(3) With $Y_1 = Y_2$ the hypergeometric function in (21) can be evaluated to give

$$G_\gamma(Y, Y) = c_\gamma \frac{(-N-1/\gamma)_N}{(-N-2/\gamma)_N}. \tag{23}$$

In this case the constant term in (19) can be taken with respect to all variables. Its value can be obtained from a special case of a theorem of Morris [8]; see also [9–12], giving agreement with (23).

It is also possible to formulate a conjecture which allows the particle density about the impurity for the state given by the wavefunction (17), with λ an even positive integer, to be calculated. By definition, if the impurity particle is at point y, then the density at point x, $\mathrm{d}(x-y)$ say, is given by

$$\mathrm{d}(x-y) = N\left(\prod_{\ell=2}^{N}\int_0^L \mathrm{d}x_\ell\right)(\phi_\lambda(y; x, x_2, \ldots, x_N)^2$$
$$\times \left[\left(\prod_{\ell=1}^{N}\int_0^L \mathrm{d}x_\ell\right)(\phi_\lambda(y; x_1, \ldots, x_N))^2\right]^{-1}. \tag{24}$$

To calculate (24) with $\lambda = 2\gamma, \gamma \in \mathbb{Z}^+$ it suffices to calculate

$$D_{\gamma,N}(Y, X_1) = CT_{\{X_2,\ldots,X_N\}} \prod_{\ell=1}^{N}\left(1-\frac{Y}{X_\ell}\right)\left(1-\frac{X_\ell}{Y}\right) \prod_{j,k=1, j\neq k}^{N}\left(1-\frac{X_j}{X_k}\right)^\gamma. \tag{25}$$

By using determinant expansions $D_{1,N}$ and $D_{2,N}$ can readily be evaluated, as too can $D_{\gamma,2}$. Furthermore, the constant term with respect to all variables in (25) follows from Morris's theorem [8]. On the basis of these exact results and some exact numerical data we propose the following result:

Conjecture 2. For $\gamma \in \mathbb{Z}^+$,

$$D_{\gamma,N}(Y,X) = d_{\gamma,N}\left\{1 + 2/(N\gamma) - \frac{1}{N\gamma}[{}_2F_1(-N,1;1-N-2/\gamma;Y/X) + {}_2F_1(-N,1;1-N-2/\gamma;X/Y)]\right\} \tag{26}$$

where

$$d_{\gamma,N} = \frac{((N-1)\gamma)!(2/\gamma)_{N-1}}{(\gamma!)^N(1/\gamma)_{N-1}}. \tag{27}$$

Finally, let us consider a multi-species classical gas with potential energy closely related to the Hamiltonian (6). The gas has potential energy

$$V_t = a^2\sum_{\alpha=1}^{t} m_\alpha \sum_{p=1}^{N_\alpha}(x_p^{(\alpha)})^2 + 2\sum_{\alpha=1}^{t}(m_\alpha)^3 \sum_{1\leqslant p<j\leqslant N_\alpha} \frac{1}{(x_p^{(\alpha)} - x_j^{(\alpha)})^2} + 2\sum_{1\leqslant\alpha<\gamma\leqslant t}\sum_{p=1}^{N_\alpha}\sum_{j=1}^{N_\gamma}\frac{m_\alpha m_\gamma(m_\alpha + m_\gamma)}{(x_p^{(\alpha)} - x_j^{(\gamma)})^2}. \tag{28}$$

This potential energy is derived by expanding

$$U_t = \sum_{\alpha=1}^{t}\frac{1}{m_\alpha}\sum_{p=1}^{N_\alpha}(u_{\alpha,p})^2 \tag{29a}$$

where

$$u_{\alpha,p} = am_\alpha x_p^{(\alpha)} - \sum_{j=1,j\neq p}^{N_\alpha}\frac{(m_\alpha)^2}{x_p^{(\alpha)} - x_j^{(\alpha)}} - \sum_{\gamma=1,\gamma\neq\alpha}^{t}\sum_{j=1}^{N_k}\frac{m_\alpha m_\gamma}{x_p^{(\alpha)} - x_j^{(\gamma)}} \tag{29b}$$

which is one of the steps required in the proof of theorem 1. Use of (9) and some further manipulation gives

$$U_t = V_t - a\left[\sum_{\alpha=1}^{t}(m_\alpha)^2 N_\alpha(N_\alpha - 1) - 2\sum_{1\leqslant\alpha<\gamma\leqslant t} m_\alpha m_\gamma N_\alpha N_\gamma\right]. \tag{30}$$

We desire to evaluate the classical partition function

$$Z_t = \left(\prod_{\alpha=1}^{t}\prod_{p=1}^{N_\alpha}\int_{-\infty}^{\infty} \mathrm{d}x_p^{(\alpha)}\right)\mathrm{e}^{-V_t/2}. \tag{31}$$

The value of Z_t can be obtained from the following conjecture:

Conjecture 3. (Special cases of this conjecture have been formulated independently by Glasser [13].) Let

$$w_j = a_{jj}x_j - \sum_{k=1,k\neq j}^{N} \frac{a_{jk}}{x_j - x_k} - \sum_{\ell=1}^{M} \frac{b_{j\ell}}{x_j - c_{j\ell}} \tag{32}$$

where

$$a_{jj} > 0, \quad a_{jk} \geqslant 0, \quad b_{j\ell} \geqslant 0, \quad c_{j\ell} \in \mathbb{R}, \quad j,k = 1,\ldots,N; \quad \ell = 1,\ldots,M. \tag{33}$$

Then for any integral function f and any parameters satisfying (33),

$$\left(\prod_{j=1}^{N}\int_{-\infty}^{\infty} \mathrm{d}x_j\right) f(w_1, w_2, \ldots, w_N)$$
$$= \left(\prod_{j=1}^{N}\frac{1}{a_{jj}}\int_{-\infty}^{\infty} dw_j\right) f(w_1, w_2, \ldots, w_N). \tag{34}$$

The conjecture (34) is true for $N = 1$ (see e.g. [14], although the result goes back to at least Boole [15] in the last century) and can easily be verified when $N = 2$ and $M = 0$. For the particular function

$$f(w_1, \ldots, w_N) = \prod_{j=1}^{N} \mathrm{e}^{-(w_j)^2/2} \tag{35}$$

with

$$M = 0, \quad a_{jj} = a \qquad \text{and} \qquad a_{jk} = g, \quad j \neq k, \quad j,k = 1,\ldots,N \tag{36}$$

the conjecture gives

$$\left(\prod_{\ell=1}^{N}\int_{-\infty}^{\infty} \mathrm{d}x_\ell\right) \exp\left(-\frac{a^2}{2}\sum_{j=1}^{N}(x_j)^2 - g^2 \sum_{1\leqslant j<k\leqslant N} \frac{1}{(x_k - x_j)^2}\right)$$
$$= \left(\frac{2\pi}{a^2}\right)^{N/2} \mathrm{e}^{-agN(N-1)/2} \tag{37}$$

which is a known result [16]. Furthermore, we have used Monte Carlo integration to verify (34), up to numerical error of 0.1%, with f given by (35) for $(N, M) = (3, 0)$ and $(2, 1)$ with various values of the parameters (33).

To evaluate (31), consider the conjecture with f given by (35) and

$$M = 0, N = \sum_{\alpha=1}^{t} N_\alpha$$
$$a_{(\alpha-1)N_\alpha+k_\alpha,(\beta-1)N_\beta+k_\beta} = \begin{cases} a m_\alpha^{1/2} & (\alpha, k_\alpha) = (\beta, k_\beta) \\ m_\alpha^{1/2} m_\beta & (\alpha, k_\alpha) \neq (\beta, k_\beta) \end{cases} \tag{38}$$

where $\alpha, \beta = 1, \ldots, t$ and $k_\alpha = 1, \ldots, N_\alpha$. Use of (30) then gives

$$Z_t = \left(\prod_{\ell=1}^{t} \left(\frac{2\pi}{a^2 m_\ell} \right)^{N_\ell/2} \right) \exp \left\{ -\frac{a}{2} \left[\sum_{\alpha=1}^{t} (m_\alpha)^2 N_\alpha (N_\alpha - 1) - \sum_{1 \leqslant \alpha < \beta \leqslant t} m_\alpha m_\beta N_\alpha N_\beta \right] \right\}. \tag{39}$$

I thank K Aomoto for sending me a copy of [1] and Doron Zeilberger for the time spent studying the original version of this letter, which led to a counterexample of the original form of conjecture 2. I acknowledge support by the Australian Research Council.

Note added. Conjecture 1 has now been proved by Zeilberger [17] and by the present author [18].

In [18], the factor $\prod_{\ell=1}^{N}(1 - Y_1/X_\ell)(1 - X_\ell/Y_2)$ of the rational function in the constant term (19) is generalized to $\prod_{\ell=1}^{N}(1 - Y_1/X_\ell)(1 - X_\ell/Y_2)(1 - Y_3/X_\ell)\ldots(1 - X_\ell/Y_m)$ and the constant term is given in terms of a generalized hypergeometric function involving Jack symmetric polynomials. Furthermore, an alternative expression for the constant term (26) in Conjecture 2 is derived. These results are all obtained as limiting cases of the evaluations of the Selberg correlation integrals given by Kaneko [19].

Conjecture 3 has now been proved by Aomoto, with the restriction $a_{jk} = a_{kj}$ in (33).

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